South East Asian J. of Mathematics and Mathematical Sciences
Vol. 21, No. 2 (2025), pp. 237-252

DOI: 10.56827/SEAJMMS.2025.2102.17 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

SOME FIXED POINT THEOREMS FOR INTERPOLATIVE MEIR-KEELER-RÉICH CONTRACTION IN A COMPLETE METRIC SPACE WITH AN APPLICATION

Steward Pastieh Pakma and Santosh Kumar

Department of Mathematics,
North Eastern Hill University,
Shillong, Meghalaya - 793022, INDIA

E-mail : stewardpakma@gmail.com, drsengar2002@gmail.com

(**Received:** *Nov. 09, 2024* **Accepted:** *Jul. 28, 2025* **Published:** *Aug. 30, 2025)*

Abstract: This paper extends the contraction condition introduced by Karapinar [8] by proposing a new interpolative Reich-Meir-Keeler-type contraction condition in a complete metric space. We establish fixed-point results for two self-mappings in a complete metric space using this condition. We provide relevant examples and an application in physical chemistry to support our findings.

Keywords and Phrases: Fixed point, common fixed point, Meir-keeler contraction, Réich-Rus-Círíc contraction, complete metric space, Interpolative contraction.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Researchers in fixed point theory have generalized the well-known Banach contraction principle [5] in various ways, enabling its application across diverse fields, including differential and integral equations, fractional differential equations, mathematical biology, physics, and computer data flow. In 1975, Marcinkiewicz [16] proved a theorem, and around the same time, Riesz and Thorin [16] developed their theorem, known as the Riesz–Thorin theorem. These foundational theorems have been instrumental in the development of interpolative theory. In 2018, Karapinar [16] introduced the concept of interpolative Kannan-type contractions. This was later extended to interpolative Hardy-Roger-type contractions by Karapinar et al. [8, 12]. Expanding on Karapinar's concepts [16], Kumar and Chilongola [12], along with Wangwe and Kumar [18], subsequently proved theorems concerning interpolative-type contractions. For more generalizations of interpolative type in different spaces, one can see [6, 8, 9, 17, 18] and the references therein.

In 1969, Meir and Keeler [13] introduced a contraction that generalized the Banach contraction, later known as the Meir-Keeler contraction. Karapinar's work [7] on interpolations involving the Meir-Keeler contraction provides insights into extending these results across various spaces and types of contractions. In 2024, Yahaya and Shagari [19] gave a very interesting result on the Meir-Keeler type contraction in a complete metric space.

In this paper, we modified Meir-Keeler contraction and Réich-Rus-Círíc contraction into a new contraction to establish the fixed point theorems in a complete metric space.

2. Preliminaries

In 1969, Meir and Keeler [13] introduced a new contraction and established a new theorem to obtain a fixed point.

Definition 2.1. [13] *Let $(\boldsymbol{Z}, \sigma)$ be a complete metric space. A mapping $P : \boldsymbol{Z} \to \boldsymbol{Z}$ is said to be a Meir-Keeler contraction on $\boldsymbol{Z}$ if for every $\epsilon > 0$, $\exists, \delta > 0$, such that*

$$\epsilon \leq \sigma(\vartheta, \nu) < \epsilon + \delta, \implies \sigma(P\vartheta, P\nu) < \epsilon, \tag{2.1}$$

for every $\vartheta, \nu \in \boldsymbol{Z}$.

Using above definition Meir-Keeler [13] proved the following theorem:

Theorem 2.1. [13] *On a complete metric space $(\boldsymbol{Z}, \sigma)$, any Meir-Keeler contraction $P : \boldsymbol{Z} \to \boldsymbol{Z}$ has a unique fixed point.*

In 2021, Karapinar [7] established a relationship between the interpolative Kannan type contraction and the Meir-Keeler contraction [12, 18], which led to a new contraction known as the Interpolative Kannan-Meir-Keeler type contraction.

Definition 2.2. *Let us consider a complete metric space $(\boldsymbol{Z}, \sigma)$. A self mapping P on $\boldsymbol{Z}$ is called interpolative Kannan-Meir-Keeler type contraction on $\boldsymbol{Z}$, if there exists $\alpha \in (0, 1)$, such that*

(i) Given, $\epsilon > 0$, there exists $\delta > 0$, such that

$$\begin{aligned} \epsilon &< [\sigma(\vartheta, P\vartheta)]^{\alpha}[\sigma(\nu, P\nu)]^{1-\alpha} < \epsilon + \delta, \\ \implies \sigma(P\vartheta, P\nu) &\leq \epsilon. \end{aligned}$$

(ii) We can rewrite the above inequality as

$$\sigma(P\vartheta, P\nu) < [\sigma(\vartheta, P\vartheta)]^{\alpha}[\sigma(\nu, P\nu)]^{1-\alpha}.$$

For all $\vartheta, \nu \in \boldsymbol{Z} \backslash Fix(P)$.
Following this definition, Karapinar [7] got a result for a fixed point.

Theorem 2.2. *For every self-interpolative Kannan-Meir-Keeler type contraction on a complete metric space, there is a fixed point.*
Reich introduced a new contraction [15] in 1971, which was known as the Réich-Rus-Círíc contraction. This contraction is a combination of the Banach and the Kannan contractions, which gave us a fixed point in a complete metric space.

Theorem 2.3. *Let* $(\boldsymbol{Z}, \sigma)$ *be a complete metric space. A mapping* $P : \boldsymbol{Z} \to \boldsymbol{Z}$ *which satisfied the condition,*

$$\sigma(P\vartheta, P\nu) \leq \alpha\sigma(\vartheta, \nu) + \beta\sigma(\vartheta, P\vartheta) + \gamma\sigma(\nu, P\nu).$$

For all $\vartheta, \nu \in \boldsymbol{Z} \backslash Fix(P)$, *where,* $\alpha, \beta, \gamma > 0$ *and* $\alpha + \beta + \gamma < 1$. *Then* P *possesses a unique fixed point.*
In 2019, Aydi et al. [3] combined the Réich-Rus-Círíc type contraction with the interpolation to get a new definition known as interpolative Réich-Rus-Círíc type contraction. This has a fixed point in a complete metric space.

Definition 2.3. [3] *Let* $(\boldsymbol{Z}, \sigma)$ *be a complete metric space. A mapping* $P : \boldsymbol{Z} \to \boldsymbol{Z}$ *is said to be an interpolative Réich-Rus-Círíc type contraction, if there are constants* $\lambda \in [0, 1)$ *and* $\alpha, \beta \in (0, 1)$, *with* $\alpha + \beta < 1$ *such that*

$$\sigma(P\vartheta, P\nu) \leq \lambda[\sigma(\vartheta, \nu)]^{\alpha}[\sigma(\vartheta, P\vartheta)]^{\beta}[\sigma(\nu, P\nu)]^{1-\alpha-\beta}, \tag{2.2}$$

For all $\vartheta, \nu \in \boldsymbol{Z} \backslash Fix(P)$.
The following result was established by using the above definition.

Theorem 2.4. [3] *Let* $(\boldsymbol{Z}, \sigma)$ *be a complete metric space. An interpolative Réich-Rus-Círíc type contraction* $P : \boldsymbol{Z} \to \boldsymbol{Z}$ *has a unique fixed point.*
For further understanding of an interpolative type contraction, we study papers [10, 11].

Definition 2.4. [14] *Let* P *and* S *be two self mappings on a non-empty set* $\boldsymbol{Z}$. *Suppose* $\nu \in \boldsymbol{Z}$ *such that* $\nu = P\vartheta = S\vartheta$, *for some* $\vartheta \in \boldsymbol{Z}$, *then* ϑ *is called a coincidence point of* P *and* S *and* ν *is called a point of coincidence of* P *and* S. *If* $\vartheta = \nu$, *then* P *and* S *have a common fixed point.*

Definition 2.5. [14] *Let* $(\boldsymbol{Z}, \sigma)$ *be a metric space. Let* $\{\vartheta_n\}$ *be a sequence in* $\boldsymbol{Z}$. *Then*

(i) A sequence $\{\vartheta_n\}$ *is said to be converges to* $\vartheta \in \boldsymbol{Z}$ *if and only if*

$$\lim_{n\to\infty} \sigma(\vartheta_n, \vartheta) = 0 \Rightarrow \lim_{n\to\infty} \vartheta_n = \vartheta.$$

(ii) A sequence $\{\vartheta_n\}$ *is said to be Cauchy sequence if and only if*

$$\lim_{n,m\to\infty} \sigma(\vartheta_n, \vartheta_m) = 0.$$

(iii) A metric space $(\boldsymbol{Z}, \sigma)$ *is said to be a complete metric space if every Cauchy sequence to an element in* $\boldsymbol{Z}$*.*

For a further understanding of a common fixed point of maps, we study papers [1, 4].

3. Main Results

To obtain the desirable results, (2.1) and (2.2) were combined into a new definition.

Definition 3.1. *Let* $(\boldsymbol{Z}, \sigma)$ *be a metric space. A mapping* $P : \boldsymbol{Z} \to \boldsymbol{Z}$ *is said to be an interpolative Réich-Rus-Círíc Meir-Keeler contraction on* $\boldsymbol{Z}$*, if there exists* $\alpha, \beta > 0$ *and* $\alpha + \beta < 1$*, such that for every* $\vartheta, \nu \in \boldsymbol{Z}\backslash Fix(P)$*, we have*

(i) Given, $\epsilon > 0$*, there exists* $\delta > 0$*, such that*

$$\epsilon < [\sigma(\vartheta, \nu)]^{\alpha}[\sigma(\vartheta, P\vartheta)]^{\beta}[\sigma(\nu, P\nu)]^{1-\alpha-\beta} < \epsilon + \delta, \quad (3.1)$$

$$\Longrightarrow \quad \sigma(P\vartheta, P\nu) \leq \epsilon. \quad (3.2)$$

(ii) We can write (3.1) and (3.2) as

$$\sigma(P\vartheta, P\nu) < [\sigma(\vartheta, \nu)]^{\alpha}[\sigma(\vartheta, P\vartheta)]^{\beta}[\sigma(\nu, P\nu)]^{1-\alpha-\beta}.$$

Using the above definition, a fixed point(s) is obtained in a complete metric space.

Theorem 3.1. *Let* $(\boldsymbol{Z}, \sigma)$ *be a complete metric space. Any interpolative Réich-Rus-Círíc Meir-Keeler contraction mapping on* $\boldsymbol{Z}$ *has a unique fixed point.*

Proof. Let $(\mathbf{Z}, \sigma)$ be a complete metric space. Let $P : \mathbf{Z} \to \mathbf{Z}$ is said to be an interpolative Réich-Rus-Círíc Meir-Keeler contraction.

Let $\vartheta_0 \in \mathbf{Z}$ be the initial point. By Picard's iteration, we can construct a sequence $\{\vartheta_n\} \in \mathbf{Z}$, so

$$\vartheta_n = P\vartheta_{n-1} = P^n\vartheta_0, \quad for\ all \quad n \in \mathbb{N}.$$

To show: $\{\sigma(\vartheta_n, \vartheta_{n+1})\}$ is a decreasing sequence.

$$\begin{aligned}
\sigma(\vartheta_n, \vartheta_{n+1}) &= \sigma(P\vartheta_{n-1}, P\vartheta_n), \\
\sigma(\vartheta_n, \vartheta_{n+1}) &< [\sigma(\vartheta_{n-1}, \vartheta_n)]^{\alpha}[\sigma(\vartheta_{n-1}, P\vartheta_{n-1})]^{\beta}[\sigma(\vartheta_n, P\vartheta_n)]^{1-\alpha-\beta}, \\
\sigma(\vartheta_n, \vartheta_{n+1}) &< [\sigma(\vartheta_{n-1}, \vartheta_n)]^{\alpha}[\sigma(\vartheta_{n-1}, \vartheta_n)]^{\beta}[\sigma(\vartheta_n, \vartheta_{n+1})]^{1-\alpha-\beta}, \\
(\sigma(\vartheta_n, \vartheta_{n+1}))^{\alpha+\beta} &< (\sigma(\vartheta_{n-1}, \vartheta_n))^{\alpha+\beta}, \\
\sigma(\vartheta_n, \vartheta_{n+1}) &< \sigma(\vartheta_{n-1}, \vartheta_n).
\end{aligned}$$

Therefore, $\{\sigma(\vartheta_n, \vartheta_{n+1})\}$ is a decreasing sequence and it will converge to some point, say 'a' i.e.

$$\lim_{n\to\infty} \sigma(\vartheta_n, \vartheta_{n+1}) = a.$$

To show: $a = 0$.
Suppose $a \neq 0$, then we can choose $\delta > 0$, such that

$$a \leq \sigma(\vartheta_n, \vartheta_{n+1}), \tag{3.3}$$

and

$$[\sigma(\vartheta_{n-1}, \vartheta_n)]^{\alpha}[\sigma(\vartheta_{n-1}, P\vartheta_{n-1})]^{\beta}[\sigma(\vartheta_n, P\vartheta_n)]^{1-\alpha-\beta} \quad < \quad a + \delta.$$

From (3.3), we have

$$\begin{aligned} a \leq& \sigma(\vartheta_n, \vartheta_{n+1}) = \sigma(P\vartheta_{n-1}, P\vartheta_n), \qquad (3.4)\\ a <& [\sigma(\vartheta_{n-1}, \vartheta_n)]^{\alpha}[\sigma(\vartheta_{n-1}, P\vartheta_{n-1})]^{\beta}[\sigma(\vartheta_n, P\vartheta_n)]^{1-\alpha-\beta} < a + \delta. \end{aligned}$$

By definition of P, we have

$$\begin{aligned} \sigma(P\vartheta_{n-1}, P\vartheta_n) &\leq a, \\ \sigma(\vartheta_n, \vartheta_{n+1}) &\leq a. \qquad (3.5) \end{aligned}$$

From (3.4) and (3.5), we have
$\sigma(\vartheta_n, \vartheta_{n+1}) = a$, which is not possible unless $a = 0$.
Therefore,

$$\lim_{n\to\infty} \sigma(\vartheta_n, \vartheta_{n+1}) = 0. \tag{3.6}$$

To show: $\{\vartheta_n\}$ is a Cauchy sequence in $\mathbf{Z}$.
Let $m, l \in \mathbb{N}$ with $l > m$, then using triangular inequality, we have

$$\begin{aligned} \sigma(\vartheta_m, \vartheta_l) &\leq \sigma(\vartheta_m, \vartheta_{m+1}) + \sigma(\vartheta_{m+1}, \vartheta_l), \\ \sigma(\vartheta_m, \vartheta_l) &\leq \sigma(\vartheta_m, \vartheta_{m+1}) + \sigma(P\vartheta_m, P\vartheta_{l-1}), \\ \sigma(\vartheta_m, \vartheta_l) &< \sigma(\vartheta_m, \vartheta_{m+1}) + [\sigma(\vartheta_m, \vartheta_{l-1})]^{\alpha}[\sigma(\vartheta_m, P\vartheta_m)]^{\beta}[\sigma(\vartheta_{l-1}, P\vartheta_{l-1})]^{1-\alpha-\beta}, \\ \sigma(\vartheta_m, \vartheta_l) &< \sigma(\vartheta_m, \vartheta_{m+1}) + [\sigma(\vartheta_m, \vartheta_{l-1})]^{\alpha}[\sigma(\vartheta_m, \vartheta_{m+1})]^{\beta}[\sigma(\vartheta_{l-1}, \vartheta_l)]^{1-\alpha-\beta}. \end{aligned}$$

From (3.6), by applying limit as $m, l \to \infty \Rightarrow \sigma(\vartheta_m, \vartheta_l) = 0$.
Therefore, $\{\vartheta_n\}$ is a Cauchy sequence in a complete metric space $\mathbf{Z}$, then there exists $\vartheta \in \mathbf{Z}$, such that

$\lim_{n\to\infty}\vartheta_n = \vartheta \Rightarrow \lim_{n\to\infty}\sigma(\vartheta_n,\vartheta) = 0$.
To show: $P\vartheta = \vartheta$.
Suppose $P\vartheta \neq \vartheta$, then $\sigma(\vartheta, P\vartheta) > 0$, so

$$\begin{aligned}\sigma(\vartheta,P\vartheta) &\le \sigma(\vartheta,\vartheta_n)+\sigma(P\vartheta_{n-1},P\vartheta),\\ \sigma(\vartheta,P\vartheta) &< \sigma(\vartheta,\vartheta_n)+[\sigma(\vartheta_n,\vartheta)]^{\alpha}[\sigma(\vartheta_n,P\vartheta_n)]^{\beta}[\sigma(\vartheta,P\vartheta)]^{1-\alpha-\beta},\\ \sigma(\vartheta,P\vartheta) &< \sigma(\vartheta,\vartheta_n)+[\sigma(\vartheta_n,\vartheta)]^{\alpha}[\sigma(\vartheta_n,\vartheta_{n+1})]^{\beta}[\sigma(\vartheta,P\vartheta)]^{1-\alpha-\beta}.\end{aligned}$$

As $n \to \infty$, we have
$\sigma(\vartheta, P\vartheta) < 0$, which is a contradiction.
Therefore, $\sigma(\vartheta, P\vartheta) = 0$,
$\Rightarrow P\vartheta = \vartheta$, i.e '$\vartheta$' is the fixed point of P.
Uniqueness of fixed point:
Let $\vartheta, \nu \in Fix(P), (if\ possible)$ in $\mathbf{Z}$ such that, $P\vartheta = \vartheta$ and $P\nu = \nu$, where $\vartheta \neq \nu$, then

$$\begin{aligned}\sigma(\vartheta,\nu) &= \sigma(P\vartheta,P\nu),\\ \sigma(\vartheta,\nu) &< [\sigma(\vartheta,\nu)]^{\alpha}[\sigma(\vartheta,P\vartheta)]^{\beta}[\sigma(\nu,P\nu)]^{1-\alpha-\beta},\\ \sigma(\vartheta,\nu) &< 0.\end{aligned}$$

which is not possible. Therefore, $\vartheta = \nu$.
Here, we will use one of the examples which satisfied the above proof.

Example 3.2. Let $\mathbf{Z} = \mathbb{N}$. Let us define a mapping $P : \mathbb{N} \to \mathbb{N}$ by
$P\vartheta = 2\vartheta - 1$, and $\sigma : \mathbb{N} \times \mathbb{N} \to [0, +\infty)$ by

$$\sigma(\vartheta,\nu) = \begin{cases} 0, \text{ if } \vartheta = \nu \\ \vartheta+\nu, \quad otherwise. \end{cases}$$

Solution: Taken $\alpha = \frac{1}{7}$, $\beta = \frac{3}{7}$ and $\vartheta = 2, \nu = 3$ so,
$\sigma(2,3) = 5$,$\sigma(P2,P3) = 8$,$\sigma(2,P2) = 5$,$\sigma(3,P3) = 8$, then with $\epsilon = 8$ and $\delta = 4$.
T is a Meir Réich-Rus-Círíc-Meir -Keeler interpolative contraction. With $\vartheta = 1$ as the unique fixed point of P.

The results due to Mohantay and Maitraz [14] serve as motivation for us to study the existence of a common fixed point between two self-maps on a complete metric space. From an established definition, a condition is put forward to check whether the common fixed point can be obtained on a complete metric space.

Theorem 3.3. *Let $(\boldsymbol{Z}, \sigma)$ be a complete metric space. Let S and P be two self mappings on $\boldsymbol{Z}$. Then, P and S have a common unique fixed point.*

Proof. Let $\vartheta_0 \in \mathbf{Z}$ be the initial point. Let $\vartheta_1 = P\vartheta_0$, $\vartheta_2 = S\vartheta_1$, $\vartheta_3 = P\vartheta_2$, $\vartheta_4 = S\vartheta_3$,...
In general, we can write the sequence,

$$\vartheta_{2n+1} = P\vartheta_{2n} \text{ and } \vartheta_{2n+2} = S\vartheta_{2n+1}, \ for\ all\ n \in \mathbb{N}.$$

Then $\{\vartheta_n\}$ be the sequence and 'σ' is defined as

$$\sigma(P\vartheta, S\nu) \leq \lambda[\sigma(\vartheta,\nu)]^{\alpha}[\sigma(\vartheta,P\vartheta)]^{\beta}[\sigma(\nu,S\nu)]^{1-\alpha-\beta},$$

where $\alpha, \beta > 0$ with $\alpha+\beta < 1$ and $\lambda \in [0,1)$. Now,

$$\begin{aligned}
\sigma(\vartheta_{2n+2},\vartheta_{2n+1}) &= \sigma(S\vartheta_{2n+1},P\vartheta_{2n}) = \sigma(P\vartheta_{2n},S\vartheta_{2n+1}),\\
\sigma(\vartheta_{2n+2},\vartheta_{2n+1}) &\leq \lambda[\sigma(\vartheta_{2n},\vartheta_{2n+1})]^{\alpha}[\sigma(\vartheta_{2n},P\vartheta_{2n})]^{\beta}[\sigma(\vartheta_{2n+1},S\vartheta_{2n+1})]^{1-\alpha-\beta},\\
\sigma(\vartheta_{2n+2},\vartheta_{2n+1}) &\leq \lambda[\sigma(\vartheta_{2n},\vartheta_{2n+1})]^{\alpha}[\sigma(\vartheta_{2n},\vartheta_{2n+1})]^{\beta}[\sigma(\vartheta_{2n+1},\vartheta_{2n+2})]^{1-\alpha-\beta},\\
(\sigma(\vartheta_{2n+2},\vartheta_{2n+1}))^{\alpha+\beta} &\leq \lambda\sigma((\vartheta_{2n},\vartheta_{2n+1}))^{\alpha+\beta},\\
\sigma(\vartheta_{2n+2},\vartheta_{2n+1}) &\leq \lambda^{\frac{1}{\alpha+\beta}}\sigma(\vartheta_{2n},\vartheta_{2n+1}),\\
\sigma(\vartheta_{2n+2},\vartheta_{2n+1}) &\leq \lambda\sigma(\vartheta_{2n},\vartheta_{2n+1}), \qquad (3.7)\\
\sigma(\vartheta_{2n+2},\vartheta_{2n+1}) &\leq \sigma(\vartheta_{2n},\vartheta_{2n+1}).
\end{aligned}$$

Therefore $\{\sigma(\vartheta_{n+2},\vartheta_{2n+1})\}$ is a decreasing sequence. From (3.7), we have

$$\begin{aligned}
\sigma(\vartheta_{n+2},\vartheta_{2n+1}) &\leq \lambda\sigma(\vartheta_{2n},\vartheta_{2n+1}) \leq \lambda^2\sigma(\vartheta_{2n-1},\vartheta_{2n}) \leq \lambda^3\sigma(\vartheta_{2n-2},\vartheta_{2n-1}),\\
\sigma(\vartheta_{n+2},\vartheta_{2n+1}) &\leq \lambda^{2n+1}\sigma(\vartheta_1,\vartheta_0). \qquad (3.8)
\end{aligned}$$

Let $m, n \in \mathbb{N}$ with $n > m$, so

$$\begin{aligned}
\sigma(\vartheta_{2m},\vartheta_{2n}) &\leq \sigma(\vartheta_{2m},\vartheta_{2m+1}) + \sigma(\vartheta_{2m+1},\vartheta_{2m+2}) + ... + \sigma(\vartheta_{2n-1},\vartheta_{2n}),\\
\sigma(\vartheta_{2m},\vartheta_{2n}) &\leq \lambda^{2m}\sigma(\vartheta_1,\vartheta_0) + \lambda^{2m+1}\sigma(\vartheta_1,\vartheta_0) + ... + \lambda^{2n-1}\sigma(\vartheta_1,x_0),\\
\sigma(\vartheta_{2m},\vartheta_{2n}) &\leq \lambda^{2m}(1+\lambda+\lambda^2+...+\lambda^{2n-2m-1})\sigma(\vartheta_1,\vartheta_0),\\
\sigma(\vartheta_{2m},\vartheta_{2n}) &\leq \frac{\lambda^{2m}}{1-\lambda}\sigma(\vartheta_1,\vartheta_0).
\end{aligned}$$

As m tends to ∞, $\lambda^{2m} \to zero$, so $\{\vartheta_{2n}\}$ is a Cauchy sequence in a complete metric space $\mathbf{Z}$. Then, there exists $\vartheta \in \mathbf{Z}$ i.e
$\lim_{n\to\infty}\vartheta_{2n} = \vartheta$.
To show: ϑ is a fixed point of P.

$$\begin{aligned}
\sigma(\vartheta,P\vartheta) &\leq \sigma(\vartheta,\vartheta_{2n}) + \sigma(P\vartheta,\vartheta_{2n}),\\
\sigma(\vartheta,P\vartheta) &\leq \sigma(\vartheta,\vartheta_{2n}) + \sigma(P\vartheta,S\vartheta_{2n_1}),\\
\sigma(\vartheta,P\vartheta) &\leq \sigma(\vartheta,\vartheta_{2n}) + \lambda[\sigma(\vartheta,\vartheta_{2n-1})]^{\alpha}[\sigma(\vartheta,P\vartheta)]^{\beta}[\sigma(\vartheta_{2n-1},S\vartheta_{2n-1})]^{1-\alpha-\beta}.
\end{aligned}$$

As n tends to ∞, we have

$$\sigma(\vartheta, P\vartheta) = 0 \Rightarrow P\vartheta = \vartheta.$$

i.e. 'ϑ' is the fixed point of P.
Uniqueness of fixed point: Let $\vartheta, \bar{\vartheta} \in Fix(P)$(if possible) i.e

$$\begin{aligned} P\vartheta &= \vartheta, \text{ and } ,P\bar{\vartheta} = \bar{\vartheta}, \\ \sigma(\vartheta, \bar{\vartheta}) &\leq \sigma(\vartheta, \vartheta_{2n}) + \sigma(\bar{\vartheta}, \vartheta_{2n}). \end{aligned}$$

As n tends to ∞, we have

$$\begin{aligned} \sigma(\vartheta, \bar{\vartheta}) &= 0, \\ \Rightarrow \vartheta &= \bar{\vartheta}. \end{aligned}$$

Let $l, k \in \mathbb{N}$ with $k > l$, so

$$\begin{aligned} \sigma(\vartheta_{2l+1}, \vartheta_{2k+1}) &\leq \sigma(\vartheta_{2l+1}, \vartheta_{2l+2}) + \sigma(\vartheta_{2l+2+1}, \vartheta_{2l+3}) + ... + \sigma(\vartheta_{2k}, \vartheta_{2k+1}), \\ \sigma(\vartheta_{2l+1}, \vartheta_{2k+1}) &\leq \lambda^{2l+1}\sigma(\vartheta_1, \vartheta_0) + \lambda^{2l+2}\sigma(\vartheta_1, \vartheta_0) + ... + \lambda^{2k}\sigma(\vartheta_1, \vartheta_0), \\ \sigma(\vartheta_{2l+1}, \vartheta_{2k+1}) &\leq \lambda^{2l+1}(1 + \lambda + \lambda^2 + ... + \lambda^{2k-2l-1})\sigma(\vartheta_1, \vartheta_0), \\ \sigma(\vartheta_{2l+1}, \vartheta_{2k+1}) &\leq \frac{\lambda^{2l+1}}{1-\lambda}\sigma(\vartheta_1, \vartheta_0). \end{aligned}$$

As l tends to ∞, $\lambda^{2l} \to 0$, so $\{\vartheta_{2l+1}\}$ is a Cauchy sequence in a complete metric space $\mathbf{Z}$. Then, there exists $\nu \in \mathbf{Z}$ i.e.

$$\lim_{l\to\infty} \vartheta_{2l+1} = \nu.$$

To show: ν is a fixed point of S.

$$\begin{aligned} \sigma(\nu, S\nu) &\leq \sigma(\nu, \vartheta_{2l+1}) + \sigma(\vartheta_{2l+1}, S\nu), \\ \sigma(\nu, S\nu) &\leq \sigma(\nu, \vartheta_{2l+1}) + \sigma(P\vartheta_{2l}, S\nu), \\ \sigma(\nu, S\nu) &\leq \sigma(\nu, \vartheta_{2l+1}) + \lambda[\sigma(\vartheta_{2l}, \nu)]^{\alpha}[\sigma(\vartheta_{2l}, P\vartheta_{2l})]^{\beta}[\sigma(\nu, S\nu)]^{1-\alpha-\beta}. \end{aligned}$$

As l tends to ∞, we have

$$\begin{aligned} \sigma(\nu, S\nu) &= 0, \\ S\nu &= \nu. \end{aligned}$$

i.e 'ν' is the fixed point of S.
Uniqueness of fixed point: Let $\nu, \bar{\nu} \in Fix(S)$(if possible) i.e

$$\begin{aligned} S\nu &= \nu \text{ and } S\bar{\nu} = \bar{\nu}, \\ \sigma(\nu, \bar{\nu}) &\leq \sigma(\nu, \vartheta_{2l+1}) + \sigma(\bar{\nu}, \vartheta_{2l+1}). \end{aligned}$$

As l tends to ∞, we have

$$\begin{aligned} \sigma(\nu, \bar{\nu}) &= 0, \\ \Rightarrow \nu &= \bar{\nu}. \end{aligned}$$

To show: $\vartheta = \nu$.

$$\begin{aligned} \sigma(\vartheta, \nu) &= \sigma(P\vartheta, S\nu) \leq \lambda[\sigma(\vartheta, \nu)]^{\alpha}[\sigma(\vartheta, P\vartheta)]^{\beta}[\sigma(\nu, S\nu)]^{1-\alpha-\beta}, \\ \Rightarrow \sigma(\vartheta, \nu) &= 0, \\ \Rightarrow \vartheta = \nu. \end{aligned}$$

Hence, P and S have a common fixed point.
We will establish the next theorem by using these results.

Theorem 3.4. *Let $(\boldsymbol{Z}, \sigma)$ be a complete metric space. Given, $\epsilon > 0$, there exists $\delta > 0$, such that for every $\vartheta, \nu \in \boldsymbol{Z} \backslash Fix(P)$ and let S and P be two self mappings on $\boldsymbol{Z}$ which satisfies*

$$\epsilon < [\sigma(\vartheta, \nu)]^{\alpha}[\sigma(\vartheta, P\vartheta)]^{\beta}[\sigma(\nu, S\nu)]^{1-\alpha-\beta} < \epsilon + \delta, \tag{3.9}$$

$$\Longrightarrow \quad \sigma(P\vartheta, S\nu) \leq \epsilon. \tag{3.10}$$

From (3.9) and (3.10), we have

$$\sigma(P\vartheta, S\nu) < [\sigma(\vartheta, \nu)]^{\alpha}[\sigma(\vartheta, P\vartheta)]^{\beta}[\sigma(\nu, S\nu)]^{1-\alpha-\beta}, \tag{3.11}$$

where $\alpha, \beta > 0$ with $\alpha + \beta < 1$. Then, P and S have a common unique fixed point.
Proof. Let $\vartheta_0 \in \mathbf{Z}$ be the initial point. Let $\vartheta_1 = P\vartheta_0$, $\vartheta_2 = S\vartheta_1$, $\vartheta_3 = P\vartheta_2$, $\vartheta_4 = S\vartheta_3$,...
In general, we can write the sequence $\{\vartheta_n\}$,

$$\vartheta_{2n+1} = P\vartheta_{2n} \text{ and } \vartheta_{2n+2} = S\vartheta_{2n+1}, \forall n \in \mathbb{N}.$$

Now, we have

$$\begin{aligned} \sigma(\vartheta_{2n+1}, \vartheta_{2n+2}) &= \sigma(P\vartheta_{2n}, S\vartheta_{2n+1}), \\ \sigma(\vartheta_{2n+1}, \vartheta_{2n+2}) &< [\sigma(\vartheta_{2n}, \vartheta_{2n+1})]^{\alpha}[\sigma(\vartheta_{2n}, P\vartheta_{2n})]^{\beta}[\sigma(\vartheta_{2n+1}, S\vartheta_{2n+1})]^{1-\alpha-\beta}, \\ \sigma(2\vartheta_{n+1}, \vartheta_{2n+2}) &< [\sigma(\vartheta_{2n}, \vartheta_{2n+1})]^{\alpha}[\sigma(\vartheta_{2n}, \vartheta_{2n+1})]^{\beta}[\sigma(\vartheta_{2n+1}, \vartheta_{2n+2})]^{1-\alpha-\beta}, \\ \sigma(\vartheta_{2n+1}, \vartheta_{2n+2})^{\alpha+\beta} &< (\sigma(\vartheta_{2n}, \vartheta_{2n+1}))^{\alpha+\beta}, \\ \sigma(2\vartheta_{n+1}, \vartheta_{2n+2}) &< \sigma(\vartheta_{2n}, \vartheta_{2n+1}). \end{aligned}$$

Therefore, $\{\sigma(\vartheta_{2n+1}, \vartheta_{2n+2})\}$ is a decreasing sequence and it will converge to some point, say 'b' i.e.

$$\lim_{n\to\infty} \sigma(\vartheta_{2n+1}, \vartheta_{2n+2}) = b.$$

To show: $b = 0$.
Suppose $b \neq 0$, then we can choose $\delta > 0$, such that

$$b \leq \sigma(\vartheta_{2n+1}, \vartheta_{2n+2}), \tag{3.12}$$

and

$$[\sigma(\vartheta_{2n}, \vartheta_{2n+1})]^{\alpha}[\sigma(\vartheta_{2n}, P\vartheta_{2n})]^{\beta}[\sigma(\vartheta_{2n+1}, S\vartheta_{2n+1})]^{1-\alpha-\beta} \quad < \quad b + \delta.$$

From (3.12), we have

$$\begin{aligned} b \leq& \sigma(\vartheta_{2n+1}, \vartheta_{2n+2}) = \sigma(P\vartheta_{2n}, S\vartheta_{2n+1}), \qquad (3.13)\\ b <& [\sigma(\vartheta_{2n}, \vartheta_{2n+1})]^{\alpha}[\sigma(\vartheta_{2n}, P\vartheta_{2n})]^{\beta}[\sigma(\vartheta_{2+1n}, S\vartheta_{2n+1})]^{1-\alpha-\beta} < b + \delta. \end{aligned}$$

From (3.9) and (3.10), we have

$$\begin{aligned} \sigma(P\vartheta_{2n}, S\vartheta_{2n+1}) &\leq b,\\ \sigma(\vartheta_{2n+1}, \vartheta_{2n+2}) &\leq b, \qquad (3.14) \end{aligned}$$

From (3.12) and (3.14), we have
$\sigma(\vartheta_{2n+1}, \vartheta_{2n+2}) = b$, which is not possible unless $b = 0$. Therefore,

$$\lim_{n\to\infty} \sigma(\vartheta_{2n+1}, \vartheta_{2n+2}) = 0. \tag{3.15}$$

To show: $\{\vartheta_{2n}\}$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ and $m < n$,

$$\sigma(\vartheta_{2m}, \vartheta_{2n}) \leq \quad \sigma(\vartheta_{2m}, \vartheta_{2m+1}) + \sigma(\vartheta_{2m+1}, \vartheta_{2m+2}) + ... + \sigma(\vartheta_{2n-1}, \vartheta_{2n}). \tag{3.16}$$

As m and n tend to $+\infty$ (3.16) tends to zero.

$$\lim_{m,n\to\infty} \sigma(\vartheta_{2m}, \vartheta_{2n}) = 0.$$

Therefore, $\{\vartheta_{2n}\}$ is a Cauchy sequence in a complete metric space $\mathbf{Z}$, then there exists $\vartheta \in \mathbf{Z}$ such that

$$\begin{aligned} \lim_{n\to\infty} \vartheta_{2n} &= \vartheta,\\ \lim_{n\to\infty} \sigma(\vartheta, \vartheta_{2n}) &= 0. \qquad (3.17) \end{aligned}$$

To show: $P\vartheta = \vartheta$.
Suppose $\vartheta \neq P\vartheta \Rightarrow \sigma(\vartheta, P\vartheta) > 0$.

$$\begin{aligned} \sigma(\vartheta, P\vartheta) &\leq \sigma(\vartheta, \vartheta_{2n}) + \sigma(P\vartheta, \vartheta_{2n}), \\ \sigma(\vartheta, P\vartheta) &\leq \sigma(\vartheta, \vartheta_{2n}) + \sigma(P\vartheta, S\vartheta_{2n-1}), \\ \sigma(\vartheta, P\vartheta) &< \sigma(\vartheta, \vartheta_{2n}) + [\sigma(\vartheta, \vartheta_{2n-1})]^{\alpha}[\sigma(\vartheta, P\vartheta)]^{\beta}[\sigma(\vartheta_{2n-1}, S\vartheta_{2n-1})]^{1-\alpha-\beta}. \end{aligned} \tag{3.18}$$

As n tends to ∞, using (3.17) and (3.18), we have

$$\sigma(\vartheta, P\vartheta) < 0,$$

which is not possible. Therefore,

$$\begin{aligned} \sigma(\vartheta, P\vartheta) &= 0, \\ P\vartheta &= \vartheta. \end{aligned}$$

Uniqueness of fixed point: Let $\vartheta, \bar{\vartheta} \in Fix(P)$(if possible) i.e

$$\begin{aligned} P\vartheta &= \vartheta, \text{ and } ,P\bar{\vartheta} = \bar{\vartheta}, \\ \sigma(\vartheta, \bar{\vartheta}) &\leq \sigma(\vartheta, \vartheta_{2n}) + \sigma(\bar{\vartheta}, \vartheta_{2n}). \end{aligned}$$

As n tends to ∞, from (3.17), we have

$$\begin{aligned} \sigma(\vartheta, \bar{\vartheta}) &= 0, \\ \Rightarrow \vartheta &= \bar{\vartheta}. \end{aligned}$$

Therefore, P have a unique fixed point.
Similarly $\{\vartheta_{2l+1}\}$ is a Cauchy sequence in $\mathbf{Z}$, then there exists $\nu \in \mathbf{Z}$, such that

$$\begin{aligned} \lim_{l\to\infty} \vartheta_{2l+1} &= \nu, \\ \lim_{l\to\infty} \sigma(\nu, \vartheta_{2l+1}) &= 0, \end{aligned} \tag{3.19}$$

and $S\nu = \nu$, where ν is a unique fixed point of S.
Suppose $\vartheta \neq \nu$, then $\sigma(\vartheta, \nu) > 0$,

$$\begin{aligned} \sigma(\vartheta, \nu) &= \sigma(P\vartheta, S\nu) < [\sigma(\vartheta, \nu)]^{\alpha}[\sigma(\vartheta, P\vartheta)]^{\beta}[\sigma(\nu, S\nu)]^{1-\alpha-\beta}, \\ \sigma(\vartheta, \nu) &< 0, \end{aligned}$$

which is not possible unless,

$$\begin{aligned} \sigma(\vartheta, \nu) &= 0, \\ \Rightarrow \vartheta &= \nu. \end{aligned}$$

Therefore, P and S have a unique common fixed point.

Example 3.5. Let $Z = \mathbb{R}^2$ and we define a metric σ as $\sigma : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ by

$$\sigma(\vartheta, \nu) = \sqrt{(\vartheta_1 - \nu_1)^2 + (\vartheta_2 - \nu_2)^2}, \text{ where } \vartheta = (\vartheta_1, \vartheta_2), \nu = (\nu_1, \nu_2) \in \mathbb{R}^2$$

and we define two mappings $P, S : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{aligned} P(\vartheta, \nu) &= (2\vartheta - 1, 3\nu + 2), \\ S(\vartheta, \nu) &= (3\vartheta - 2, 2\nu + 1). \end{aligned}$$

By taking $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{3}$, we get $(\vartheta, \nu) = (1, -1)$ as its common unique fixed point of P and S.

4. Application

Depending upon the dimension of a space, molecules have a certain number of degrees of freedom with regard to their motions, like translational, vibrational and rotational motions, with respect to the axes. An equipartition theorem says that the average energy at temperature T is either used to contribute to a molecule's movement or energy is distributed to the movement of a molecule with each degree of freedom. The total (internal) energy of molecules is given by

$$U_m(T) = U_m(0) + nRT, \tag{4.1}$$

where n varies, depending on whether the molecule is a gas ($n = \frac{3}{2}$), a linear molecule ($n = \frac{5}{2}$) or a nonlinear molecule ($n = 0$). $U_m(0)$ is the molar internal energy at $T = 0$, R is a constant. In motion state, even if $T = 0$, the internal energy may not be zero.

Theorem 4.1. *When a non-constant temperature T from (4.1) is applied to a molecule under some space (may be a closed surface), then (4.1) has a solution under a defined metric*

$$\sigma(U_m(T), U_m(S)) < [\sigma(T, S)]^{\alpha}[\sigma(T, U_m(T))]^{\beta}[\sigma(S, U_m(S))]^{1-\alpha-\beta}, \tag{4.2}$$

where $\alpha, \beta > 0$ and $\alpha + \beta < 1$.

Proof. Let us define $\sigma(x, y) = ||x - y||$.

Here,

$$\begin{aligned} \sigma(U_m(T), U_m(S)) &= ||U_m(T) - U_m(S)||, \\ \sigma(U_m(T), U_m(S)) &= ||nRT - nRS||, \\ \sigma(U_m(T), U_m(S)) &= n|R| \cdot ||T - S||. \end{aligned}$$

Suppose,

$$\begin{aligned} ||U_m(T)-U_m(S)| &\geq ||T-S||^{\alpha}||T-U_m(T)||^{\beta}||S-U_m(S)||^{1-\alpha-\beta},\\ n|R|\cdot||T-S|| &\geq ||T-S||^{\alpha}||T-U_m(T)||^{\beta}||S-U_m(S)||^{1-\alpha-\beta},\\ n|R|\cdot||T-S||^{1-\alpha} &\geq ||T-U_m(T)||^{\beta}||S-U_m(S)||^{1-\alpha-\beta}. \end{aligned} \tag{4.3}$$

If $T=S$, from (4.3),

$$\begin{aligned} 0 &\geq ||T-U_m(T)||^{\beta}||T-U_m(T)||^{1-\alpha-\beta},\\ 0 &\geq ||T-U_m(T)||^{1-\alpha},\\ 0 &= ||T-U_m(T)||^{1-\alpha}, \qquad (4.4)\\ U_m(T) &= T. \end{aligned}$$

This implies that$U_m(T)$ has a unique fixed (point) temperature. From (4.1) we have,

$$\begin{aligned} U_m(T)-T &= U_m(0)+nRT-T,\\ U_m(T)-T &= U_m(0)+(nR-1)T,\\ \Rightarrow ||U_m(T)-T|| &= ||U_m(0)+(nR-1)T||. \end{aligned}$$

From (4.4)

$$\begin{aligned} 0 &= ||U_m(0)+(nR-1)T||,\\ 0 &= U_m(0)+(nR-1)T,\\ (nR-1)T &= -U_m(0),\\ T &= \frac{U_m(0)}{1-nR}. \end{aligned}$$

So, T is a constant temperature, which is a contradiction. Therefore,

$$\begin{aligned} n|R|\cdot||T-S|| &\leq ||T-S||^{\alpha}||T-U_m(T)||^{\beta}||S-U_m(S)||^{1-\alpha-\beta},\\ \Rightarrow \sigma(U_m(T),U_m(T)) &< [\sigma(T,S)]^{\alpha}[\sigma(T,U_m(T))]^{\beta}[\sigma(S,U_m(S))]^{1-\alpha-\beta}. \end{aligned}$$

Hence, $U_m(T)$ has a solution.

5. Conclusion

In this paper, we obtain a unique fixed point by interpolating two contractions, namely the Meir-Keeler contraction and the Reich-Ciric contraction, on a complete metric space with a single mapping. This was supported by an example shown earlier. The condition established earlier with regard to a common fixed point for double mappings was supported by Theorem 3.3 and Theorem 3.4. This was also supported by an example given earlier.

References

[1] Abbas M., Jungck G., Common fixed point results for noncommuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl., 341.1, (2008), 416-420.

[2] Atkins P., de Paula J., Physical Chemistry, Eighth edition, Oxford University Press, 2006.

[3] Aydi H., Chen Chi-Ming, Karapinar E., Interpolative Ćirić-Reich-Rus Type Contractions via the Branciari Distance, Mathematics, 7(1) (2019), 84.

[4] Azam A., Arshad M., Beg I., Common fixed point theorems in cone metric spaces, J. Nonlinear Sci. Appl., 2(4) (2009), 204-213.

[5] Banach S., Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fundam. Math, 3 (1922), 133–181.

[6] Gautam P., Singh S. R., Kumar S. and Verma S., On nonunique fixed point theorems via interpolative Chatterjea type Suzuki contraction in quasi-partial b-metric space, J. Math., Vol. 2022 (2022), Article ID 2347294.

[7] Karapınar E., Interpolative Kannan- Meir-Keeler type contraction, Advances in the Theory of Nonlinear Analysis and its Application, 5(4) (2021) 611–614.

[8] Karapinar E., Revisiting the Kannan type contractions via interpolation, Advances in the Theory of Nonlinear Anal. Appl., Vol. 2(2) (2018), 85–87.

[9] Karapinar E., Alqahtani O., and Aydi H., On interpolative Hardy-Rogers type contractions, Symmetry, Vol. 11(1)(2019), 8.

[10] Karapınar E., Aydi H., Mitrovic Z. D., On interpolative Boyd-Wong and Matkowski type contractions, TWMS J. Pure Appl. Math., 11, (2) (2020), 204-212.

[11] Karapınar E., On interpolative metric spaces, Filomat, 38(22) (2024), 7729-7734.

[12] Kumar S. and Chilongola J., Fixed-Point Theorems for $\omega-\psi$ Hardy-interpolative Hardy-Rogers-Suzuki type contraction in a compact Quasi partial b-Metric space, Journal of Function Spaces, Volume 2023, Article ID 3911534, 12 pages.

[13] Meir A. and Keeler E., A theorem on contraction mapping, J. Math. Anal. Appl., 28(1969), 326-329.

[14] Mohantay S. K., Maitraz R., Coincidence Points and Common Fixed Points for Expansive Type Mappings in b-Metric Spaces, Iran. J. Math. Sci. Inform., 11(1) (2016), 101-113.

[15] Reich S., Some remarks concerning contraction mappings, Canad. Math. Bull. Vol. 14 (1), (1971).

[16] Sette R. M. M., Fernandez D. L., da Silva E. B., A Theory for Interpolation of Metric Spaces, Axioms, 13(7) (2024), 439.

[17] Wangwe L., Kumar S., A Common Fixed Point Theorem for Generalised F-Kannan Mapping in Metric Space with Applications, Abst. Appl. Anal., Vol. 2021, (2021) Article ID 6619877, 12 pages.

[18] Wangwe L. and Kumar S., Fixed point Results for Interpolative ψ-Hardy-Rogers type contraction mappings in quasi-partial b-metric space with some Applications, J. Anal., (2022) 18 Pages.

[19] Yahaya S., Shagari M. S., Fixed Point of Hybrid Jaggi-Meir-Keeler Type Multivalued Contraction, Appl. Appl. Math., 19(2), (2024), 1-13.